CAPM
AND
The Markowitz Efficient Frontier

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1 Basic concepts and a short introduction

The beta coefficient is used to measure how an asset moves by comparing it to the performance of the stock market.

By default, many investors choose to compare the performance of the stock to the performance of the S&P 500 since this is generally a reliable indicator of how the market is moving. In fact, we will be analysing the correlation of the performance of the assets to the performance of the S&P 500.

An intuitive understanding of some of the different values β can take are as follow. If:

1. $\beta = 1$, the value of the stock is very correlated to the value of the S&P 500.

2. $\beta = 0$, the stock is not correlated to the S&P 500. In other words, if the value of the S&P 500 varies, the value of our asset does not necessarily vary.

3. $\beta = -1$, the price of the stock will **most often** (but not always) go in the opposite direction to the price of the S&P 500.

4. $-1 < \beta < 1$, sometimes a change in the price of the stock follows a change in the price of the S&P 500, sometimes it doesn't. For example, throughout a year, a stock with a β in this range might move in the same direction as the S&P 500 during February and in the opposite direction to the S&P 500 during October.

Things start getting more interesting when we start considering $\beta > 1$ or $\beta < -1$.

When $|\beta| > 1$, the value of the asset tends to be more volatile than the price of the S&P 500. Therefore, one can start playing with these values, as well as values of $|\beta| < 1$ to determine a desired level of risk one wants to take, since the beta of a portfolio (β_p) with n assets, is given by:

$$\beta_p = \Sigma_{i=1}^n w_i * \beta_i$$

for:

$$\Sigma_{i=1}^n w_i = 1$$

where w_i is the percentage of the whole portfolio dedicated to asset i.

Consequently, this is a very simplistic way to measure the risk of a portfolio.

2 CAPM

The formula for the β of an asset is as follows:

$$\beta = \frac{Covariance(R_s, R_m)}{Variance(R_m)}$$

here, we define R_s as the return of the asset and R_m as the return on the overall market. The covariance between R_m and R_s is the directional relationship between the returns on two assets. Thus, covariance is what determines the sign of β. Also, we can see that the greater the covariance, the greater the value of β, thus the risk that we are taking is larger.

We can now explore at the Capital Asset Pricing Model (CAPM), which will show us a direct application of the value of β.

2.1 What is the CAPM?

The CAPM is a mathematical model that describes the relationship between systematic risk and expected return for assets, particularly stocks.

The formula is as follows:

$$E_a = R + \beta(E_m - R)$$

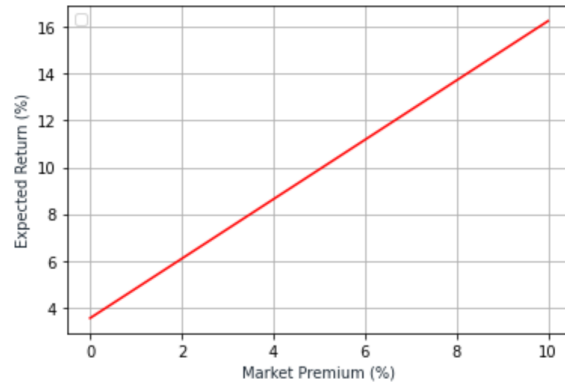Here, E_a is the expected return on our asset.

Figure 1: Expected return against Market Premium

R is the risk-free interest rate, in other words the interest we get back from putting our money in a secure bank and getting interest out of it or from buying government bonds. These sort of investments are considered to be 'safe' because theoretically, most big banks or stable governments will eventually be able to pay investors back.

For American stocks, we usually take this value to be the 10-year government bond yield, which is as of nowadays, 3.55%.

Finally, E_m is the expected return on the market, we will further explore this idea in an example.

For simplicity, we will use the notation M_p for the value $E_m - R$ and we will refer to it as the 'Market risk premium', which is simply the extra return an investor would expect from holding an asset with a higher risk than our risk less asset, i.e. 10-year government bonds. In fact, note how this term becomes zero if we invest on government bonds, showing that the risk an investor assumes is none.

What this formula tells us is that for a given market premium ($E_m - R$ in our formula), which is a measure of the risk an investor assumes when buying a security, we get an expected return as an output.

To get a better grasp of the meaning of our model, we will look at an example.

Suppose we want to get the CAPM graph for the stock of Apple. We can get the value of β from Yahoo Finance, who use monthly data over a 36-month period to measure the effect (if any) that a change in the S&P 500 has on the price of the Apple stock. From this source, we get that the β of Apple is 1.27.

We can now plot a graph on Python with M_p on the x-axis and the expected return on the y-axis. This plot can be seen in Figure 1.

Now, we have to estimate the market premium we are interested on. We already know what R is, so we have to estimate the expected return on the market, E_m. This is down to personal interpretations and predictions.

For the sake of our example, we will assume for 2023 to be a bullish year, so we'll take $E_m = 7.55\%$. Therefore, the expected return we are interested on is the point on the y-axis when our market premium equals $7.55\% - 3.55\% = 4\%$, by plugging in this value into our formula for E_r, we get that the expected return for the Apple stock is 8.63%.

This value can be useful for many reasons, for instance when putting the pieces together to decide the asset allocation for an investment portfolio, which is the next topic of this paper.

3   The Efficient Frontier

Now that we have a better understanding of our model for expected returns, we would like to stretch our knowledge by studying the Markowitz Efficient Frontier (or simply the Efficient Frontier), which is one of the pillars of the Modern Portfolio Theory.

We will start by giving off a formula for the 'Sharpe ratio' of an asset, which will be necessary for our understanding of the model.

It is given by:

$$S_a = \frac{E_a - R}{\sigma_a}$$

where σ_a is the standard deviation of the asset's excess return and S_a is the Sharpe Ratio.

Basically, the Sharpe Ratio measures how much excess return an investment or portfolio earns per unit of risk taken, relative to the risk-free rate of return. The **higher** the Sharpe Ratio, the better the risk-adjusted performance of the investment or portfolio.

We can connect this idea to the Efficient Frontier, which is a graph that describes the optimal set of portfolios that provide the highest expected return for a given level of risk, or the lowest level of risk for a given expected return. The optimal portfolio on the Efficient Frontier is defined to be the one with the highest Sharpe Ratio.

It is important to highlight that for graphing purposes, it is more convenient to plot the logarithm of the expected returns because they have several desirable properties that make them more convenient to work with. One of the main properties is that log-returns are symmetric around zero, meaning that positive and negative returns are treated equally. This is important because in financial modeling, we often need to consider the possibility of both gains and losses.

In order to understand the Efficient Frontier in a more practical way, we shall look at an example.

Suppose that after doing some research, we came up with a list of four assets we would like to invest on and they are: Microsoft, Google, Tesla and Amazon. From this, we can program some code in Python that gives off the expected **log-returns** (and the standard deviations) of hundreds of portfolios with these assets, keeping in mind that each portfolio has a unique asset allocation associated to it.

This simulation gives us the graph in Figure 2 (where each dot represents a portfolio). As we can see, the upper side of the Efficient Frontier (which is the hyperbola), gives us the 'best' asset allocation for a given level of expected volatility, i.e. risk of the investment.

Likewise, the little red dot represent the asset allocation with the largest Sharpe Ratio. In this example, we get that we would ideally allocate 0.3% of our resources to Microsoft, 26.7% to Google, 48% to Tesla and 25% to Amazon.

As mentioned before, this is one of many ways to construct an investment portfolio, but none of them will be effective if we don't know the market well enough and if we haven't done enough research.

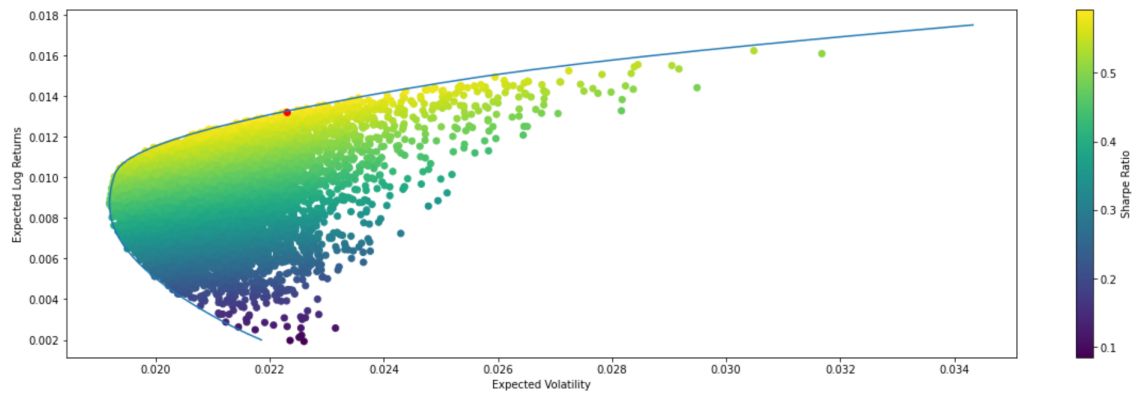If you would like to further explore this last Python simulation, send me an email to my Warwick email and I will be more than happy to share the code with you and guide you through it.

Figure 2: Efficient Frontier